Simple Calculations of Classical Spin Angular Momentum

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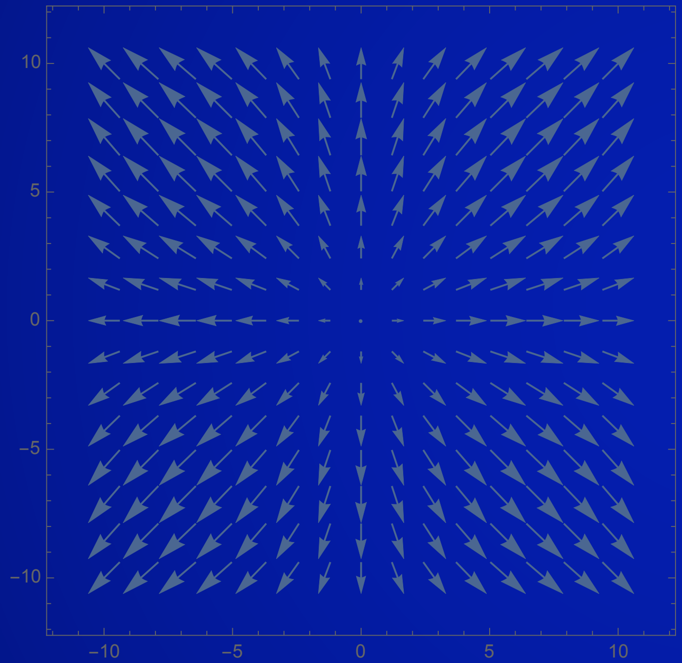
Introduction

Spin angular momentum is ordinary angular momentum:

- It represents the effort (torque * time) required to stop rotation of an inertial substance.
- It is the conjugate momentum to angular velocity.
- **It is independent of coordinates.**

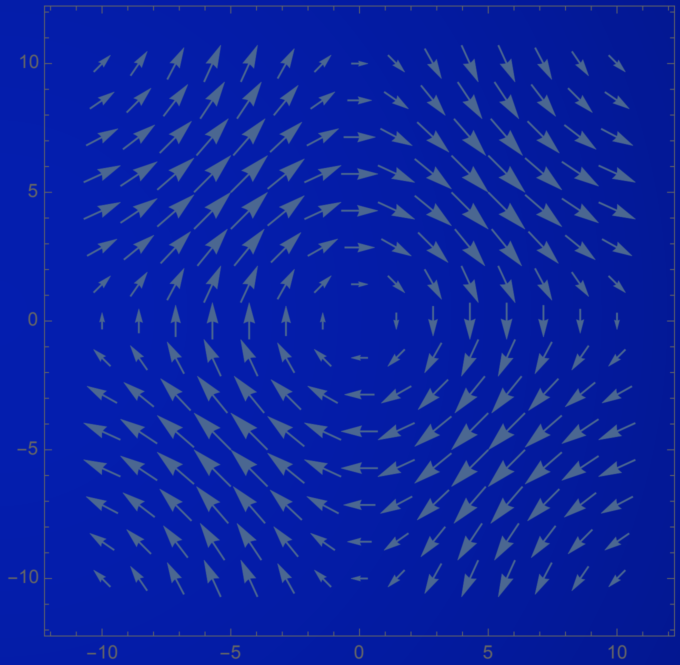
Spin Density: Motion Fields

Fundamentally, motion has two types:



Irrotational

$$\nabla \times \mathbf{u} = 0$$



Incompressible (Rotational)

$$\nabla \cdot \mathbf{u} = 0$$

Spin Density: Incompressible Motion

Helmholtz Decomposition of momentum density:

$$\mathbf{p} \equiv \rho\mathbf{u} = \mathbf{p}_0 + \nabla\Phi + \frac{1}{2}\nabla \times \mathbf{s}$$

Incompressible and in "rest" frame:

$$\nabla\Phi = 0,\ \mathbf{p}_0 = 0$$

$$\nabla \cdot \mathbf{u} = 0 \rightarrow \rho = \text{constant}$$

$$\nabla \cdot \rho\mathbf{u} = \nabla \cdot \mathbf{p} = 0$$

$$\mathbf{p} = \frac{1}{2}\nabla \times \mathbf{s}$$

Spin Density

$$\mathbf{p} = \frac{1}{2}\nabla \times \mathbf{s}$$

- Spin density is the vector field whose curl is equal to twice the incompressible momentum density ($\mathbf{p} = \rho\mathbf{u}$).[1]
- Valid for classical & quantum physics.

[1] R. A. Close, "A classical Dirac bispinor equation," in *Ether Space-time & Cosmology, vol. 3*, edited by M. C. Duffy and J. Levy (Apeiron, Montreal, 2009).

Dirac Equation

Electron:

$$\partial_t \psi + c\,\gamma^5 \sigma_{\boldsymbol{j}} \partial_j \psi + iM\gamma^0 \psi = 0$$

Multiply $\psi^\dagger \sigma_{\boldsymbol{i}}$ and add adjoint:

$$\psi^\dagger \sigma_{\boldsymbol{i}} \partial_t \psi + \psi^\dagger c\,\gamma^5 \sigma_{\boldsymbol{i}} \sigma_{\boldsymbol{j}} \partial_j \psi + adj. = 0$$

Spin density equation (multiply $\hbar/2$)

$$0 = \partial_t[\psi^\dagger \boldsymbol{\sigma} \psi] + c\nabla[\psi^\dagger \gamma^5 \psi]$$
$$-ic[\nabla\psi^\dagger \times \boldsymbol{\sigma}\psi + \psi^\dagger \gamma^5 \boldsymbol{\sigma} \times \nabla\psi]$$

Same as elastic solid linear shear waves.

$$0 = \partial_t^2 \mathbf{Q} - c^2\nabla(\nabla \cdot \mathbf{Q}) + c^2\nabla \times (\nabla \times \mathbf{Q})$$

$$\mathbf{s} = \partial_t \mathbf{Q}; \quad \boldsymbol{\xi} = (1/2)\nabla \times \mathbf{Q}; \quad \mathbf{u} = \partial_t \boldsymbol{\xi}$$

Spin Density

Elastic waves include dynamic & conjugate momenta:

$$\mathbf{P}_{\boldsymbol{Total}} = -\mathrm{Re}\left[\psi^{\dagger} i \nabla \psi\right] + \frac{1}{2}\nabla \times \left[\psi^{\dagger}\frac{\boldsymbol{\sigma}}{\mathbf{2}}\psi\right]$$

$$= P_{wave} + \rho\mathbf{u}$$

$$\mathbf{J} = -\mathrm{Re}\left(\mathbf{r} \times \left[\psi^{\dagger} i \nabla \psi\right]\right) + \left[\psi^{\dagger}\frac{\boldsymbol{\sigma}}{\mathbf{2}}\psi\right]$$

$$= \mathbf{L}_{\boldsymbol{wave}} + \mathbf{s}$$

Separation of **J** and **P** into two terms *"has a direct physical meaning only for physical agencies that are endowed with inertia"* (Rosenfeld) i.e. spin ⇒ aether

*L´eon Rosenfeldr. Sur le tenseur dimpulsion-nergie (on the energy-momentum tensor). Mmoires Acad. Roy. de Belgique, 18:1–30, 1940. (Translated by D. H. Delphenich)

Spin Density

Compare transverse wave on a string
($\rho_1 = \text{mass/length},\ \mathrm{T} = \text{tension}$):

$$\rho_1 \partial_t^2 \xi - \mathrm{T} \partial_x^2 \xi = 0$$

Momentum per unit length:

$$P_{\boldsymbol{Total}} = \rho_1 (\partial_t \xi)(\partial_x \xi) + \rho_1 \partial_t \xi$$

$$= P_{wave} + \rho_1 u$$

Spin Density

Caution:

Spin density is <u>classical</u> angular momentum density: due to rotation of an inertial substance. But <u>quantum mechanical</u> motion of material objects is wave motion. What we normally call "rotation of an inertial substance" is actually "wave" or "orbital" angular momentum.

I refer to "classical" spin density as a property of the vacuum or "aether" in a classical physics interpretation of relativistic quantum mechanics.

Spin Density

- Two types of momentum:
  "intrinsic" & "wave" (or "dynamic")
- Two types of angular momentum:
  "intrinsic" ($\mathbf{s}$) & "wave" ($\mathbf{L}$).
- $\mathbf{L}$ is due to potential energy & torque.
- Spin $\mathbf{s}$ and $\mathbf{p} = \rho\mathbf{u} = (1/2)\nabla \times \mathbf{s}$ are ordinary momenta.
- Objects propagate through space as waves. We observe wave momenta.

Spin Density vs Magnetostatics

$$\mathbf{u} = \frac{1}{2\rho}\nabla\times\mathbf{s}; \quad \mathbf{w} = \frac{1}{2}\nabla\times\mathbf{u}$$

$$\mathbf{B} = \nabla\times\mathbf{A}; \quad \mathbf{J}_M = \frac{4\pi}{c}\nabla\times\mathbf{B}$$

Relations between spin density (**s**), velocity (**u**), and angular velocity (**w**) are similar to relations between magnetostatic vector potential (**A**), magnetic field (**B**), and current density ($\mathbf{J}_M$).

Integrated Spin Density

$$\mathbf{p} = \rho\mathbf{u} = \frac{1}{2}\nabla \times \mathbf{s}; \quad \mathbf{w} = \frac{1}{2}\nabla \times \mathbf{u}$$

$$\mathbf{S} = \int \mathbf{r} \times \mathbf{p}\, \boldsymbol{d^3r} = \int \mathbf{s}\, \boldsymbol{d^3r} + \mathrm{b.\,t.}$$

$$K = \int \frac{1}{2}\rho u^2\, d^3r = \int \frac{1}{2}\mathbf{w} \cdot \mathbf{s}\, \boldsymbol{d^3r} + \mathrm{b.\,t.}$$

Spin density yields the same total intrinsic angular momentum and rotational kinetic energy as conventional calculations.

Spin Density: Conjugate Momentum

For Lagrangian density $\mathfrak{L} = \frac{1}{2}\rho u^2$,

$$\frac{\delta}{\delta \mathbf{w}} \int \frac{\rho u^2}{2}\, d^3 r = \frac{\delta}{\delta \mathbf{w}} \int \frac{\mathbf{w} \cdot \mathbf{s}}{2}\, \boldsymbol{d^3 r} = \int \mathbf{s}\, d^3 r$$

s is the momentum conjugate to angular velocity (**w**).

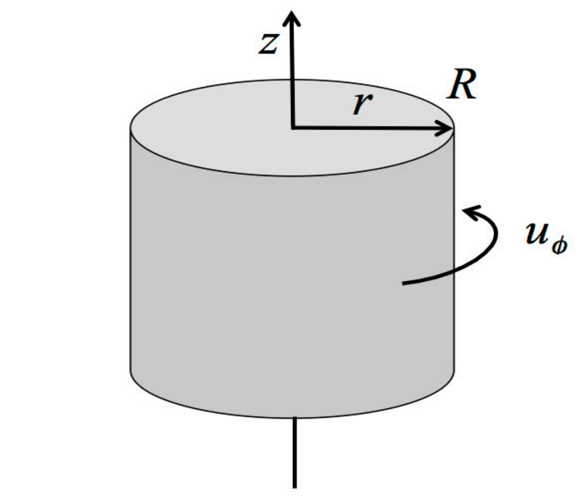
p is the momentum conjugate to velocity (**u**).

Spin Density

$$\mathbf{p} = \frac{1}{2}\nabla \times \mathbf{s}$$

- For <u>incompressible</u> motion, **s**, **p**, and **w** have similar relationships as magnetostatic **A**, **B**, and current **J**.
- But no one calculates “total” integrated vector potential **A**.

Rotating Cylinder



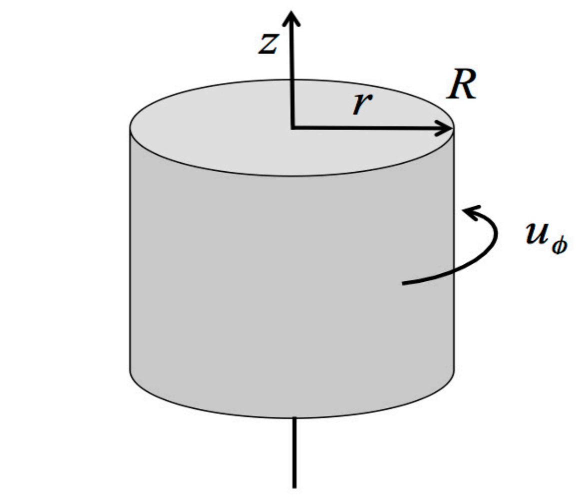
Parabolic radial distribution of spin density.

$$s_z = \rho w_0 [R^2 - r^2] \qquad r \leq R$$

$$u_\phi = \frac{1}{2\rho}\frac{\partial}{\partial r} s_z = r w_0 \qquad r \leq R$$

$$w_z = \frac{1}{2r}\frac{\partial}{\partial r}(r u_\phi)$$
$$= w_0[1 - R\delta(R-r)/2] \qquad r \leq R$$

Rotating Cylinder



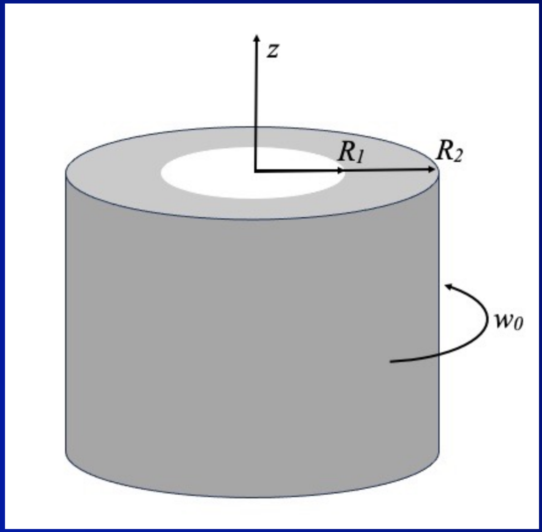
Integration yields the usual total angular momentum and kinetic energy.

$$S_z = \int \rho w_0 [R^2 - r^2]\, d^3r$$

$$= 2\pi\rho H \left(\frac{R^4}{4}\right) w_0 = \frac{MR^2}{2} w_0 = I w_0$$

$$K = \int \frac{1}{2} \mathbf{w} \cdot \mathbf{s}\, d^3r = \frac{1}{2} I w_0^2$$

Rotating Annular Ring



Parabolic radial distribution of spin density.

$$s_z = \rho w_0[R_2^2 - R_1^2] \qquad r < R_1$$

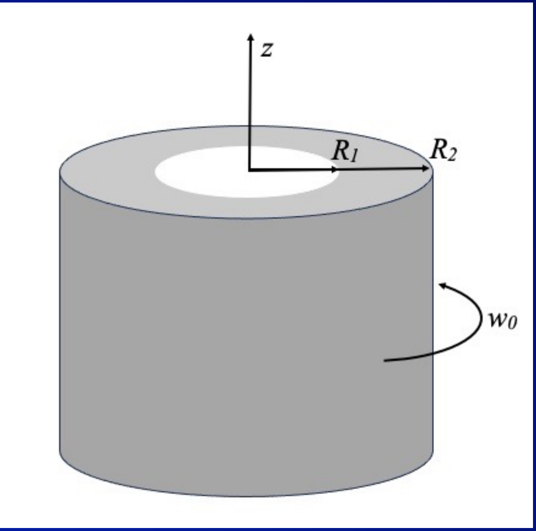
$$s_z = \rho w_0[R_2^2 - r^2] \qquad R_1 \le r \le R_2$$

$$u_\phi = -\frac{1}{2\rho}\frac{\partial}{\partial r}s_z = r w_0 \qquad R_1 \le r \le R_2$$

$$w_z = \frac{1}{2r}\frac{\partial}{\partial r}(r u_\phi) = w_0 \qquad R_1 < r < R_2$$

Rotating Annular Ring

Integration yields the usual total angular momentum and kinetic energy.

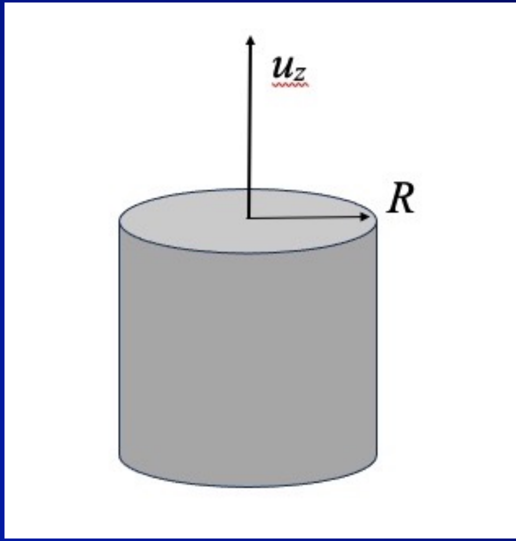


$$S_z = \int s_z \, d^3r = \frac{\rho w_0}{2}\pi(R_2^4 - R_1^4)H$$

$$= M w_0 \left(\frac{R_2^2 + R_1^2}{2}\right) = I w_0$$

$$K = \int \frac{1}{2}\mathbf{w}\cdot\mathbf{s}\, d^3r = \frac{1}{2} I w_0^2$$

Translating Cylinder



Nonzero velocity only inside cylinder.

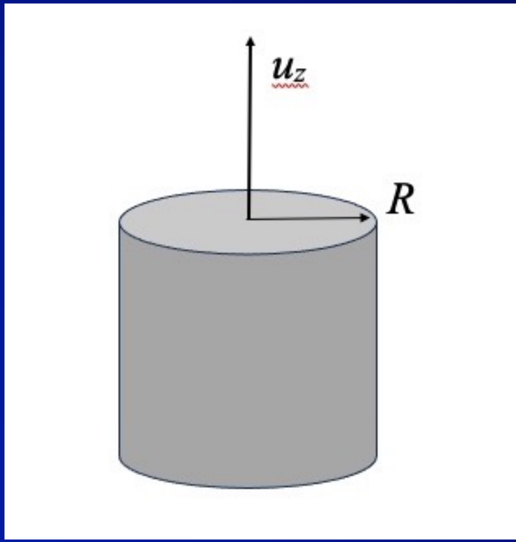
$$s_\phi = \rho u_z r \qquad r \le R$$

$$s_\phi = \rho u_z R^2/r \qquad r > R$$

$$u_z = \frac{1}{2\rho r}\partial_r(r s_\phi) = u_z \qquad r \le R$$

$$\mathrm{w}_\phi = -\frac{1}{2}\partial_r u_z = \frac{1}{2}u_z\delta(r-R)$$

Translating Cylinder



Integration yields the usual total angular momentum and kinetic energy.

$$\mathbf{S} = \int s_\phi \widehat{\boldsymbol{\phi}}\, d^3r = 0$$

$$K \;=\; \int \frac{1}{2} \mathbf{w} \cdot \mathbf{s}\, d^3r$$

$$= \int \frac{1}{2} \left(\frac{1}{2} u_z \delta(r - R) \right) (\rho u_z r)\, d^3r$$

$$= \frac{1}{2} \rho u_z^2 (\pi R^2\, H) = \frac{1}{2} M u_z^2$$

Summary

- Spin angular momentum is ordinary angular momentum in coordinate-independent form.
- It represents the effort (torque * time) required to stop rotation of an inertial substance.
- It is related to “moment of momentum” through integration by parts.
- It is the conjugate momentum to angular velocity.
- Spin density, velocity, and angular velocity have similar relations as magnetostatic vector potential, magnetic field, and current.

Publications

- Torsion Waves in Three Dimensions: Quantum Mechanics With a Twist," **Found. Phys. Lett.** 15(1):71-83, Feb. 2002.
- "A Classical Dirac Bispinor Equation," in *Ether Space-time & Cosmology, vol. 3,* eds. M. C. Duffy & J. Levy, (Aperion, Montreal, 2009).
- "Exact Description of Rotational Waves in an Elastic Solid," **Adv. App. Clifford Algebras 21**:273-281, 2010.
- "The Mirror Symmetry of Matter and Antimatter," **Adv. App. Clifford Algebras 21**:283-295, 2010.
- *The Wave Basis of Special Relativity* (Verum Versa, 2014, 2023)
- "Spin Angular Momentum and the Dirac Equation," **Electr. J. Theor. Phys. 12**(33):43-60, 2015

- More at: www.ClassicalMatter.org