# Introduction to wave mechanics: Interactions

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The first-order quantum mechanical Dirac equation is interpreted as a representation of a second-order vector wave equation for spin angular momentum density (or spin density). This interpretation offers students a simple classical physics interpretation of relativistic quantum mechanics. This paper outlines how a classical wave theory of spin density can be used to describe particle-like waves and their interactions, offering students a conceptual bridge between classical physics and quantum mechanics. Wave interference of spin eigenfunctions gives rise to the Pauli exclusion principle and electromagnetic potentials. Classical interpretations of magnetic flux quantization and the Coulomb potential are presented. A classical version of the single-electron Lagrangian for quantum electrodynamics is also presented.

Keywords: classical interpretation, Dirac equation, elastic solid, magnetic flux quantization, quantum electrodynamics, quantum mechanics pedagogy, spin angular momentum, spin density, teaching quantum mechanics, wave mechanics

## 1. INTRODUCTION

Previous work has proposed that physics students should be introduced to quantum mechanics using the Dirac equation, which is both relativistic and describes spin angular momentum, rather than via the Schrödinger equation. Classical spin density ($\mathbf{s}$) is the field whose curl is equal to twice the momentum density ($\mathbf{p}$). The relationship to Dirac wave functions is: [1–3]

$$\mathbf{s} = \partial_t \mathbf{Q} \equiv \frac{1}{2}\left[\psi^\dagger \boldsymbol{\sigma} \psi\right]; \tag{1a}$$

$$c\nabla\cdot\mathbf{Q} \equiv -\frac{1}{2}\left[\psi^\dagger \gamma^5 \psi\right]; \tag{1b}$$

$$c^2\left\{\nabla\times\nabla\times\mathbf{Q}\right\} \equiv -\frac{\mathrm{i}c}{2}\left\{\left[\nabla\psi^\dagger\right]\times\gamma^5\boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma}\times\nabla\psi\right\}; \tag{1c}$$

$$0 = \frac{\mathrm{i}c}{2}\nabla\cdot\left\{\left[\nabla\psi^\dagger\right]\times\gamma^5\boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma}\times\nabla\psi\right\}\ . \tag{1d}$$

The Dirac formalism, in addition to quantum mechanical application, has a variety of applications in describing classical physics. [1–11] There has also been observation of quantum mechanical behavior of classical systems. Quantum statistics have been observed in experiments using silicone droplets bouncing on a vibrating water tank. [12–18] Such experiments provide a physical realization of Bohmian mechanics, or pilot-wave theory. [19–21]

This work examines how wave interactions give rise to the Pauli exclusion principle and electromagnetic potentials. We discuss magnetic flux quantization and magnetic flux and electric charge of an electron. Finally, we relate the classical Lagrangian to that of quantum electrodynamics.

## 2. WAVE INTERACTIONS

Suppose we have two Dirac wave functions $\psi_A$ and $\psi_B$, representing particle-like waves $A$ and $B$. Adding the wave functions yields a total wave function $\psi_T$ satisfying:

$$\begin{aligned}\psi_T^\dagger\boldsymbol{\sigma}\psi_T &= (\psi_A+\psi_B)^\dagger\boldsymbol{\sigma}(\psi_A+\psi_B)\\ &= \psi_A^\dagger\boldsymbol{\sigma}\psi_A + \psi_B^\dagger\boldsymbol{\sigma}\psi_B + \psi_A^\dagger\boldsymbol{\sigma}\psi_B + \psi_B^\dagger\boldsymbol{\sigma}\psi_A\,.\end{aligned} \tag{2}$$

Since the spins must be additive, the total wave function is not generally the sum of the individual wave functions. However, we can treat the wave functions as being independent if the interference terms cancel [1]. This cancelation imposes a vector constraint on the wave functions:

$$\psi_A^\dagger\boldsymbol{\sigma}\psi_B + \psi_B^\dagger\boldsymbol{\sigma}\psi_A = 0\,. \tag{3}$$

Since spin density has three components, there are three independent constraints on the wave interaction. In other words, one wave function interacts with another wave function in three different ways. This paper will only consider electromagnetic forces.

Assuming either of the waves to be a spin eigenfunction everywhere, one component of this constraint requires the wave functions to anti-commute:

$$\psi_A^\dagger\psi_B + \psi_B^\dagger\psi_A = 0\,. \tag{4}$$

For waves representing identical particles, this is the Pauli exclusion principle. Hence we can conclude that standing waves described by spin eigenfunctions are fermions.

The anti-commutation of wave functions is not true in general, but we can force the cancellation by introducing a phase shift at each point between the two wave functions. Such phase shifts have no effect on the actual dynamics of the total wave, but allow us to pretend that each particle wave maintains its separate identity even though there is actually only one combined wave. Of course, this procedure is only valid if the particles interact weakly enough to remain distinguishable during the interaction. This limitation does not invalidate the basic premise that physical quantities are fully determined by the spin density field.

The phase shift ($\delta$) is determined from the constraint:

$$\mathrm{Re}(\psi_A^\dagger \exp(\mathrm{i}\delta)\psi_B) = 0, \tag{5}$$

or

$$\mathrm{Re}(\psi_A^\dagger \psi_B)\cos\delta - \mathrm{Im}(\psi_A^\dagger \psi_B)\sin\delta = 0\,. \tag{6}$$

This yields:

$$\tan\delta = \frac{\mathrm{Re}(\psi_A^\dagger \psi_B)}{\mathrm{Im}(\psi_A^\dagger \psi_B)}\,. \tag{7}$$

If we define $\exp(\mathrm{i}\beta) = \psi_A^\dagger \psi_B / |\psi_A^\dagger \psi_B|$, then:

$$\tan\delta = \cot\beta\,. \tag{8}$$

and:

$$\exp(\mathrm{i}\delta) = \pm\mathrm{i}\exp(-\mathrm{i}\beta) = \pm\mathrm{i}\,\frac{\psi_B^\dagger \psi_A}{|\psi_A^\dagger \psi_B|}\,. \tag{9}$$

Therefore the phase angles are related by:

$$\delta = \frac{\pi}{2} - \beta \pm n\pi \tag{10}$$

where $n$ is an integer. Note that $\delta$ is only unique within an arbitrary multiple of $\pi$. The $\pi/2$ phase shift is a constant, so we will ignore it while analyzing phase shifts of individual particles.

Suppose we start with two wave functions $\psi'_A$ and $\psi'_B$, initially non-overlapping and normalized to one ($\hbar$ will be multiplied explicitly to provide the dimension of angular momentum). We will assume that each particle is shifted by a phase attributable to the other particle: $\psi_A = \psi'_A \exp(\mathrm{i}\delta_A)$ and $\psi_B = \psi'_B \exp(\mathrm{i}\delta_A)$, where the primed variables have zero interference. As they approach each other, the total wave function is $\psi_T = \psi_A + \psi_B$. The phase shifts satisfy:

$$\exp(\mathrm{i}(\delta_A - \delta_B)) = \mathrm{i}\,\frac{\psi_B^\dagger \psi_A}{|\psi_A^\dagger \psi_B|} \equiv \exp(\mathrm{i}(\varphi_B - \varphi_A + \pi/2)) \tag{11}$$

where $\varphi_A$ and $\varphi_B$ are phases associated with particles $A$ and $B$, respectively. A simple choice would be to weight the phase shifts by the relative magnitudes of each wave:

$$\begin{aligned}\delta_A &= \frac{|\psi_B|^2}{|\psi_A|^2 + |\psi_B|^2}(\varphi_B - \varphi_A + \pi/2)) \\ \delta_B &= -\frac{|\psi_A|^2}{|\psi_A|^2 + |\psi_B|^2}(\varphi_B - \varphi_A + \pi/2))\end{aligned} \tag{12}$$

Since the $\pi/2$ phase shift is constant, we will omit it from the following analysis.

The original wave functions satisfy the free-particle wave equation, e.g.

$$\hbar\partial_t(\exp[-\mathrm{i}\delta_A]\psi_A) = -\mathrm{i}H_0 \exp[-\mathrm{i}\delta_A]\psi_A. \tag{13}$$

We take $\psi'_A$ to be an electron wave function with free particle hamiltonian $H_0\psi'_A = (-c\gamma^5\boldsymbol{\sigma}\cdot\mathrm{i}\hbar\nabla + m_e c^2\gamma^0)\psi'_A$.

Expanding the Dirac equation for $\psi_A$ yields:

$$\left(\hbar(\mathrm{i}\partial_t + [\partial_t\delta_A]) + \hbar c\gamma^5\boldsymbol{\sigma}\cdot(\mathrm{i}\nabla + [\nabla\delta_A]) - m_e c^2\gamma^0\right)\psi_A = 0\,, \tag{14}$$

where square brackets indicate that the derivatives apply only to the variables inside the brackets.

The modified Hamiltonian is:

$$H\psi_A = \mathrm{i}\hbar\partial_t\psi_A = -\hbar[\partial_t\delta_A]\psi_A + c\gamma^5\boldsymbol{\sigma}\cdot\hbar(-\mathrm{i}\nabla - [\nabla\delta_A])\psi_A + m_e c^2\gamma^0\psi_A\,. \tag{15}$$

The modified momentum density is:

$$\psi_A^\dagger \exp[\mathrm{i}\delta_A](-\mathrm{i}\hbar\nabla)\exp[-\mathrm{i}\delta_A]\psi_A = \psi_A^\dagger(-\mathrm{i}\hbar\nabla - \hbar\nabla\delta_A)\psi_A \tag{16}$$

We will drop the subscript $A$ in the following analysis.

The mass term in the Dirac equation, which may be interpreted as rotation of wave velocity, does not appear in the second-order vector wave equation for spin density. Therefore, it is clear that changes in wave velocity may be accomplished via linear processes.

The wave force density, or time derivative of momentum density, is found from the partial time derivative and the commutator of the hamiltonian with the momentum operator:

$$\begin{aligned} d_t P_i &= \psi_A^\dagger(-\hbar\partial_t\partial_i\delta_A + \frac{\mathrm{i}}{\hbar}[H,(-\mathrm{i}\hbar\partial_i - \hbar\partial_i\delta_A)])\psi_A \\ &= -\psi^\dagger\left[\hbar\partial_t\partial_i\delta_A - \hbar\partial_i(\partial_t\delta_A) - \hbar c\partial_i(\gamma^5\sigma_j\partial_j\delta_A) + \hbar c(\gamma^5\sigma_j\partial_j)\partial_i\delta_A\right]\psi \\ &= \psi^\dagger\left[\hbar(\partial_i\partial_t - \partial_t\partial_i)\delta_A - \hbar c\gamma^5\sigma_j(\partial_i\partial_j - \partial_j\partial_i)\delta_A)\right]\psi \end{aligned} \tag{17}$$

Note that there is no change in momentum density if the derivatives commute everywhere. Hence plane waves would not be affected by the presence of other plane waves. However, because the phases are multivalued, the derivatives do not generally commute.

To simplify analysis, we define the vector potential by $q\mathbf{A} \equiv -\hbar\nabla\delta_A$, the electric potential by $q\Phi = \partial_t\delta_A$, the charge density by $\rho_A \equiv q\psi_A^\dagger\psi_A$, and the current density by $\mathbf{J}_A \equiv \psi_A^\dagger qc\gamma^5\boldsymbol{\sigma}\psi_A$. The rate of change of momentum density is:

$$d_t P_i = \rho_A(-\partial_i\Phi_A - \partial_t A_i) + J_j(\partial_i A_j - \partial_j A_i) \tag{18}$$

In vector form:

$$d_t\mathbf{P} = \rho_A(-\nabla\Phi - \partial_t\mathbf{A}) + \mathbf{J}_A \times (\nabla\times\mathbf{A})\,. \tag{19}$$

This is equivalent to the Lorentz force with:

$$\mathbf{E} = -\nabla\Phi - \partial_t\mathbf{A}\,, \tag{20a}$$

$$\mathbf{B} = \nabla\times\mathbf{A}\,. \tag{20b}$$

Others have similarly identified the vector potential $\mathbf{A}$ as the gradient of a multivalued field.[22–24] The curl of such gradients need not be identically zero. This interpretation is also consistent with Synge's "primitive quantization" in which Planck's constant $h$ represents the action for a single wave cycle. [25]

Suppose wave $B$ is a macroscopic wave ($|\psi_B|^2 >> |\psi_A|^2$) with azimuthal dependence $\psi_B \sim \exp(-\mathrm{i}m_B\phi_B/2)$, where $\phi_B$ is the azimuthal angle around the local axis $z_B$ through the center of particle $B$, and $m_B$ is the azimuthal angular quantum number. The phase shift of $\psi_A$ would then vary by $-m_B\pi$ along a closed path around the $z_B$-axis:

$$\oint \nabla\delta_A\cdot d\boldsymbol{\ell} = m_\phi\pi \tag{21}$$

for some integer $m_\phi$. Stoke's law then yields quantization of magnetic flux:

$$\iint \mathbf{B}\cdot\hat{\mathbf{n}}\,dS = \oint \mathbf{A}\cdot d\ell = m_\phi\pi\frac{\hbar}{q} = m_\phi\frac{h}{2q}\,. \tag{22}$$

This classical quantization of magnetic flux is consistent with de Broglie's observation in a 1963 interview that "... in quantum phenomena one obtains quantum numbers, which are rarely found in mechanics but occur very frequently in wave phenomena and in all problems dealing with wave motion." [26]

### 2.1. Electron Interactions

Alternatively, suppose that $\psi_B$ has a phase factor $\exp(\mathrm{i}(m_B\phi_B - \omega_B t))$, and similarly for $\psi_A$. This phase factor is appropriate for a vector spherical harmonic wave. The angular frequency of the bispinor wave function is related to mass by $\hbar\omega_B = m_e c^2$. The angular frequency of the vector wave function would be twice this.

For simplicity, we define $r_B = |\mathbf{r} - \mathbf{r}_B|$ to be the distance from the center of particle $B$ at $\mathbf{r}_B$. The distance $r_A$ is similarly defined for particle $A$. The distance between the centers of the two particles is $r_{AB}$.

The phase shift of $\psi_A$ is:

$$\delta_A = \frac{|\psi_B|^2}{|\psi_A|^2 + |\psi_B|^2}[(m_B\phi_B - \omega_B t)) - (m_A\phi_A - \omega_A t))] \tag{23}$$

### 2.2. Magnetic Flux

The computed magnetic flux around $B$ will vary with distance from the particle center. We are interested in the phase shift of $A$ attributable to particle $B$, near the center of $A$. Near $r_A = 0$ we have $|\psi_A|^2 >> |\psi_B|^2$. Hence we can approximate the phase shift as:

$$\delta_A \approx \left( \left[ \frac{|\psi_B|^2}{|\psi_A|^2} \right]_{\mathbf{r} \approx \mathbf{r}_A} \right) (m_B \phi_B - \omega_B t) \tag{24}$$

Assume that each charge density has a peak value $\rho_M$ and asymptotic radial decay of $\rho_M \alpha/(\kappa r)$.

$$\delta_A \approx \left( \frac{\alpha}{\kappa r_B} \right) (m_B \phi_B - \omega_B t) \tag{25}$$

This is the phase shift in the vicinity of $r_B \approx r_{AB}$ where $|\psi_A|^2 >> |\psi_B|^2$. The magnetic vector potential is:

$$\frac{e}{\hbar}\mathbf{A} = -\left( \frac{\alpha}{\kappa r_B} \right) \left( \frac{m_B}{r_B \sin(\theta_B)} \right) \hat{\boldsymbol{\phi}}_B + \frac{\alpha}{\kappa r_B^2} (m_B \phi_B - \omega_B t) \hat{\mathbf{r}} \tag{26}$$

Magnetic flux is due to the azimuthal component of the vector potential:

$$\frac{e}{\hbar}\mathbf{A} = -\left( \frac{\alpha}{\kappa r_B} \right) \left( \frac{m_B}{r_B \sin(\theta_B)} \right) \hat{\boldsymbol{\phi}}_B \tag{27}$$

Given the expression for $\mathbf{A}$, the effective magnetic field at finite distance $r_B$ from the center of $B$ is:

$$\mathbf{B} = \frac{\hbar \, \alpha m_B}{e \kappa r_B^3 \sin(\theta_B)} \hat{\boldsymbol{\theta}}_B \tag{28}$$

This is not a simple dipole field.

For $m_B = 1/2$, the magnetic flux through the plane $\sin(\theta_B) = 1$ is:

$$\oint (\mathbf{A} \cdot d\ell) = -\pi - \frac{\hbar \alpha}{e \kappa r_B} \pi \tag{29}$$

Taking $\alpha = e^2/(4\pi\epsilon_0 \hbar c)$ to the be fine structure constant and $\kappa = m_e c/\hbar$, the radial variation of this flux is the same as for an electron.

### 2.3. Electric Field

To calculate an electric force, we note that the phase velocity for a stationary spherical harmonic wave is:

$$\mathbf{v}_p = \frac{2\omega_B}{m_B} r \sin\theta_B \hat{\boldsymbol{\phi}} \tag{30}$$

For simplicity, we neglect any motion of particle $B$. Changes of phase are attributable to convection. Using $\partial_t \delta_A = -\mathbf{v}_p \cdot \nabla \delta_A$ yields:

$$eE_i = \hbar((\partial_i v_j)\partial_j + v_j(\partial_i \partial_j - \partial_j \partial_i))\delta_A \tag{31}$$

The second term cannot contribute to the field because the phase velocity is zero at the point where the derivatives do not commute. The radial component is thus:

$$eE_r = \hbar(\partial_r v_\phi) A_\phi \tag{32}$$

Direct substitution yields:

$$eE_r = \frac{\hbar \omega_B \, \alpha}{\kappa r_B^2} \tag{33}$$

Letting $\kappa = \omega_B/c$ yields:

$$\mathbf{E} = \frac{\hbar c \, \alpha}{e r_B^2} \hat{\mathbf{r}} \tag{34}$$

This is the usual electric field of a point charge located at $r_B = 0$.

Recall that we neglected any contribution from the phase of $A$. Since the electric field is radial, any contribution from the phase shift proportional to $\varphi_A$ would not affect the bulk motion of particle $A$.

### 2.4. Maxwell's Equations

The electromagnetic fields defined above are also subject to Maxwell's equations. The definitions of $\mathbf{E}$ and $\mathbf{B}$ imply Faraday's Law and Gauss' magnetic law:

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}\,, \tag{35a}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{35b}$$

Gauss' electric law and Ampere's law define the charge and current densities ($\rho_e$ and $\mathbf{J}$, respectively):

$$\nabla \cdot \mathbf{E} = -(\nabla \cdot \partial_t \mathbf{A} + \nabla^2 \Phi) \equiv \frac{\rho_e}{\epsilon_0}\,, \tag{36a}$$

$$\begin{aligned} \nabla \times \mathbf{B} - \frac{1}{c^2}\partial_t \mathbf{E} &= \nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2}(\partial_t^2 \mathbf{A} + \partial_t \nabla \Phi) \\ &\equiv \mu_0 \mathbf{J}\,. \end{aligned} \tag{36b}$$

As derived above, the electric field of a spherical harmonic wave corresponds to that of a point charge. These definitions of charge and current densities are consistent with the continuity equation:

$$\partial_t \rho_e + \nabla \cdot \mathbf{J} = 0\,. \tag{37}$$

There is a discrepancy between the point-like source charge density of Maxwell's equations, and the object charge density of wave mechanics. In wave mechanics, the object charge density is "smeared out" throughout the wave function. It is similarly smeared out in quantum mechanics, in which case the smearing is attributed to uncertainty of position.

Hence particle-like waves in an elastic solid can behave like fermions, with electromagnetic potentials derived from phase shifts that result from wave interference.

### 2.5. Quantum Electrodynamics

It is customary in quantum mechanics textbooks to define $\bar{\psi} \equiv \psi^\dagger \gamma^0$, replace $\psi^\dagger$ with $\bar{\psi}\gamma^0$, and define the "4-vector" of matrices $\gamma^\mu \equiv (\gamma^0, \gamma^0\gamma^5\boldsymbol{\sigma})$. The 4-potential is $A_\mu = (\Phi, -\mathbf{A})$ and the 4-current $(\rho, \mathbf{J})$ is $J^\mu = q\bar{\psi}\gamma^\mu\psi$. These changes of variables are intended to make the theory look more "relativistic". It is also common to use "natural" units with $\mu_0 = \epsilon_0 = c = 1$. Using this notation with $\partial_\mu = (\partial_t, \nabla)$, the Lagrangian density for two interacting electrons is:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + \bar{\psi}_B[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_B]\psi_B\,. \tag{38}$$

Separating the interaction of particle $B$ yields:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + \bar{\psi}_B[\gamma^\mu(\mathrm{i}\partial_\mu) - m_B]\psi_B - J^\mu A_\mu\,. \tag{39}$$

Since the Dirac equation is satisfied for each particle, this is equivalent to:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + J^\mu A_\mu - J^\mu A_\mu\,. \tag{40}$$

Relationships between potentials and sources are given in Eqs. 36. Assuming time-independence with zero divergence of the vector potential and zero curl of the electric field, the sources become:

$$\frac{\rho_e}{\epsilon_0} = -\nabla^2 \Phi\,, \tag{41a}$$

$$\mu_0 \mathbf{J} = \nabla \times (\nabla \times \mathbf{A})\,. \tag{41b}$$

Therefore:

$$J^\mu A_\mu = -\Phi \nabla^2 \Phi - \mathbf{A} \cdot (\nabla \times \nabla \times \mathbf{A})\,. \tag{42}$$

According to Green's first identity:

$$-\int_V \Phi \nabla^2 \Phi dV = \int (\nabla \Phi)^2 dV - \int_{\partial V} \Phi \mathbf{n} \cdot \nabla \Phi dS\,. \tag{43}$$

Similarly:

$$-\int_V \mathbf{A}\cdot(\nabla\times\nabla\times\mathbf{A})dV = -\int_V (\nabla\times\mathbf{A})^2 dV + \int_{\partial V}\mathbf{A}\times(\nabla\times\mathbf{A})dS\,. \tag{44}$$

Using the definitions of **E** and **B** while neglecting boundary integrals yields:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + (E^2 - B^2) - J^\mu A_\mu \tag{45}$$

This differs from the (non-quantized) Lagrangian density of quantum electrodynamics (QED) by a factor of 1/2 in front of $(E^2 - B^2)$. This difference is resolved by the fact that when varying the potentials $A_\mu$, the source densities $J^\mu$ should be regarded as functions of $A_\mu$. However, it is conventional to vary the potentials independently of the source densities, yielding only half of the correct value. When computing variations of $E^2$ and $B^2$, both factors in $E^2$ (and $B^2$) are varied. To eliminate this double-counting and be consistent with independent variation of the potentials, a factor of 1/2 must be introduced:

$$\mathcal{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m_A]\psi_A + \frac{1}{2}(E^2 - B^2) - J^\mu A_\mu \tag{46}$$

This is the Lagrangian density of non-quantized QED, in which a single charged fermion interacts with an electromagnetic field. Generalization to multiple interacting particles requires a quantization procedure with raising and lowering operators to allow for changes in the numbers of particles.

## 3. DISCUSSION

We have outlined a similarity between a classical model of interacting waves in an elastic solid to quantum electrodynamics (QED). This interpretation of QED, and by extension the Standard Model, is that it represents a decomposition of the classical spin density field into interacting elementary particles. Others have also associated quantum mechanical behavior with waves in an elastic solid. [10, 27–30]

## 4. CONCLUSIONS

This paper describes interactions of classical waves of spin density. Wave interference of spin eigenfunctions gives rise to the Pauli exclusion principle and electromagnetic potentials, with suggested interpretations of magnetic flux quantization and the Coulomb potential. The Lagrangian density of single-fermion quantum electrodynamics is also given a classical physics interpretation. Hence classical wave theory offers insight into the physical basis for relativistic quantum mechanics.


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