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Dirac Operators for Kinetic and Potential Energy

Robert Close, PhD.

Clark College (retired)
robert.close@classicalmatter.org

Spin Density

Belinfante-Rosenfeld symmetric stress-energy tensor implies dynamic (wave) & conjugate momenta:

$$\mathbf{P}_{Total} = -\mathrm{Re}\left[\psi^\dagger i\nabla\psi\right] + \frac{1}{2}\nabla\times\left[\psi^\dagger\frac{\boldsymbol{\sigma}}{\mathbf{2}}\psi\right]$$

$$= \mathbf{P}_{wave} + \frac{1}{2}\nabla\times\mathbf{s}$$

$$\mathbf{J} = -\mathrm{Re}\left(\mathbf{r}\times\left[\psi^\dagger i\nabla\psi\right]\right) + \left[\psi^\dagger\frac{\boldsymbol{\sigma}}{\mathbf{2}}\psi\right]$$

$$= \mathbf{L}_{wave} + \mathbf{s}$$

Elastic waves in a solid also have dynamic & conjugate (or "intrinsic") momenta.

Classical Spin Density

Incompressible Helmholtz decomposition:

$$\mathbf{p} \equiv \rho\mathbf{u} = \cancel{\mathbf{p}_0 + \nabla\Phi} + \frac{1}{2}\nabla \times \mathbf{s}$$

Angular momentum:

$$\mathbf{S} = \int \mathbf{r} \times \mathbf{p}\, d^3r = \int \mathbf{s}\, d^3r \cancel{+ \text{b.t.}}$$

Kinetic energy: $\left(\mathbf{w} = \frac{1}{2}\nabla \times \mathbf{u}\right)$

$$K = \int \frac{1}{2}\rho u^2\, d^3r = \int \frac{1}{2}\mathbf{w}\cdot\mathbf{s}\, d^3r \cancel{+ \text{b.t.}}$$

Momentum conjugate to **w**: $(\mathfrak{L} \sim K)$

$$\frac{\partial \mathfrak{L}}{\partial \mathbf{w}} = \frac{\mathbf{s}}{2} + \frac{\mathbf{s}}{2} = \mathbf{s}$$

Equation of Evolution

Momentum density: $\partial_t \mathbf{p} + \mathbf{u} \cdot \nabla \mathbf{p} = \mathbf{f}$

Changes due to convection & force.

Spin density: $\partial_t \mathbf{s} + \mathbf{u} \cdot \nabla \mathbf{s} - \mathbf{w} \times \mathbf{s} = \boldsymbol{\tau}$

Changes ~ convection, rotation, & torque.

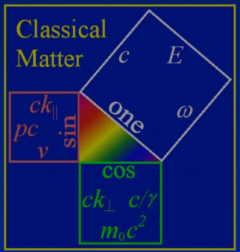
$\mathbf{s} \equiv \partial_t \mathbf{Q}$, Displacement: $\boldsymbol{\xi} = \frac{1}{2\rho} \nabla \times \mathbf{Q}$

Elastic solid: Torque = $\boldsymbol{\tau} = -\frac{\mu}{\rho} \nabla \times \nabla \times \mathbf{Q}$

$$\partial_t^2 \mathbf{Q} + \mathbf{u} \cdot \nabla \partial_t \mathbf{Q} - \mathbf{w} \times \partial_t \mathbf{Q} = c^2 \nabla^2 \mathbf{Q}$$

Nonlinearity $\Rightarrow$ soliton solutions

Dirac Equation



Linear vector wave equations:

$$0 = \partial_t[\psi^\dagger \boldsymbol{\sigma}\psi] + c\nabla[\psi^\dagger\gamma^5\psi]$$
$$-ic[\nabla\psi^\dagger \times \gamma^5\boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma} \times \nabla\psi]$$

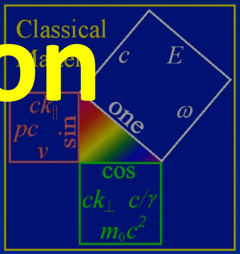
$$0 = \partial_t^2\mathbf{Q} - c^2\nabla(\nabla\cdot\mathbf{Q}) + c^2\nabla\times\nabla\times\mathbf{Q}$$
$$= \partial_t^2\mathbf{Q} - c^2\nabla^2\mathbf{Q}$$

$$\mathbf{s} = \partial_t\mathbf{Q} = (1/2)\psi^\dagger\boldsymbol{\sigma}\psi$$

$$c\nabla\cdot\mathbf{Q} = -(1/2)\psi^\dagger\gamma^5\psi$$

$$c\nabla\times\nabla\times\mathbf{Q} = -\frac{i}{2}[\nabla\psi^\dagger \times \gamma^5\boldsymbol{\sigma}\psi + \psi^\dagger\gamma^5\boldsymbol{\sigma}\times\nabla\psi]$$

Nonlinear Equation of Evolution



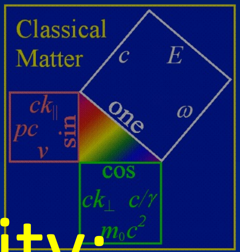
Nonlinear Dirac eqn. $\boldsymbol{s} \equiv \partial_t \mathbf{Q} = \left[\psi^\dagger \frac{\boldsymbol{\sigma}}{2} \psi\right]$:

$$\partial_t \psi = -c\,\gamma^5 \boldsymbol{\sigma} \cdot \nabla\psi - \mathbf{u} \cdot \nabla\psi - \frac{\mathrm{i}}{2} \breve{w}_0\,\gamma^0 \psi - \frac{\mathrm{i}}{2} \mathbf{w} \cdot \boldsymbol{\sigma}\psi$$

Propagation, convection, rotation of wave velocity, rotation of the solid medium.

Mass represents rotation of wave velocity.*

*c.f. Hestenes, *Found. Phys.* **20**(10):1213-32,1990



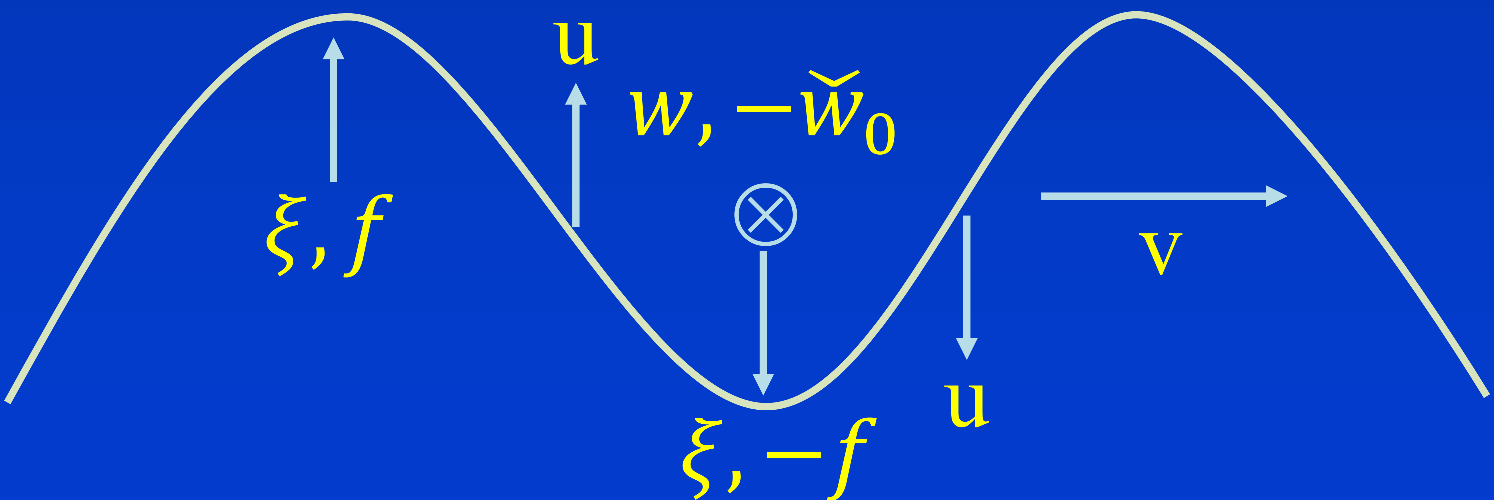
Potential Energy

$\psi^{\dagger}\gamma^{0}\psi$ ~ displacement. $\breve{w}_0$ ~ force density:

$$\breve{w}_0 = \frac{1}{2\rho c} f = \frac{1}{4\rho} \nabla^2 \left(\psi^{\dagger} \frac{\gamma^0}{2} \psi \right)$$

PE density is: $U = \frac{\breve{w}_0}{4} \psi^{\dagger} \gamma^0 \psi \sim -\frac{1}{2} \mathbf{f} \cdot \boldsymbol{\xi}$

Rotational PE: $U_R = \mathcal{E} - \mathrm{U}$



Hamiltonian Density

$$\mathcal{H} = \mathrm{Re}\left(-c\psi^\dagger\gamma^5\boldsymbol{\sigma}\cdot i\nabla\psi - \mathbf{u}\cdot\psi^\dagger i\nabla\psi\right) + \frac{\breve{w}_0}{4}\psi^\dagger\gamma^0\psi + \frac{\mathbf{w}}{4}\cdot\psi^\dagger\boldsymbol{\sigma}\psi$$

$$= \left(\mathcal{E} - 0 + \frac{1}{2}\mathbf{f}\cdot\boldsymbol{\xi}\right) + \frac{1}{2}\mathbf{w}\cdot\mathbf{s}$$

$$= U_R + K_R$$

Rotational potential energy: $U_R = \mathcal{E} + \frac{1}{2}\mathbf{f}\cdot\boldsymbol{\xi}$

$$= \mathrm{Re}\left(-c\psi^\dagger\gamma^5\boldsymbol{\sigma}\cdot i\nabla\psi\right) + \frac{\breve{w}_0}{4}\psi^\dagger\gamma^0\psi$$

Rotational kinetic energy: $K_R = \frac{1}{2}\mathbf{w}\cdot\mathbf{s} = \mathcal{E} - \frac{1}{2}\rho u^2$

$$= \frac{1}{4}\mathbf{w}\cdot\psi^\dagger\boldsymbol{\sigma}\psi$$

Example: Plane Wave

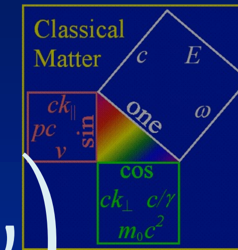
$$\psi = \sqrt{\frac{\omega Q_0}{2}} \begin{bmatrix} 1 \\ \cos(\omega t - kz) - i\,\dfrac{\omega Q_0 k^2}{4\rho c}\sin(\omega t - kz) \\ \cos(\omega t - kz) - i\,\dfrac{\omega Q_0 k^2}{4\rho c}\sin(\omega t - kz) \\ 1 \end{bmatrix}$$

$$s_x = \frac{1}{2}\psi^\dagger \sigma_x \psi = \omega Q_0 \cos(\omega t - kz)$$

$$s_y = \frac{1}{2}\psi^\dagger \sigma_y \psi = 0$$

$$s_z = \frac{1}{2}\psi^\dagger \sigma_z \psi = 0$$

Example: Plane Wave



$$U_R = \left(-\mathrm{Re}\left(\psi^\dagger\gamma^5\boldsymbol{\sigma}\cdot i\nabla\psi\right) + \frac{1}{4}\breve{w}_{\mathbf{0}}\psi^\dagger\gamma^0\psi\right)$$

$$= \frac{k^2\omega^2Q_0^2}{8\rho}\sin^2(\omega t - kz)$$

$$K_R = \frac{1}{4}\mathbf{w}\cdot\boldsymbol{\sigma}\psi = \frac{k^2\omega^2Q_0^2}{8\rho}\cos^2(\omega t - kz)$$

$$i\partial_t\psi = \left(-\gamma^5\boldsymbol{\sigma}\cdot i\nabla\psi + \frac{1}{2}\breve{w}_{\mathbf{0}}\gamma^0\psi\right) + \frac{1}{2}\mathbf{w}\cdot\boldsymbol{\sigma}\psi$$

$$\mathcal{E} = \mathbf{v}\cdot\mathbf{P} - 2U + 2K_R$$

RHS $\neq$ Hamiltonian (because nonlinear).

QM sets $\mathbf{v}\cdot\mathbf{P} = 0$ and $m_ec^2 = 2U$.

Operators as Generators

$$\mathbf{P}\psi = -i\nabla\psi + \frac{1}{2}\nabla \times \frac{\boldsymbol{\sigma}}{2}\psi$$

$$\mathbf{J}\psi = -\mathbf{r} \times i\nabla\psi + \frac{\boldsymbol{\sigma}}{2}\psi$$

- **J** is generator of rotations (orbital for coordinate, spin for direction)
- **P** is generator of translations.
    - Wave momentum shifts coordinate ($\mathbf{r}$).
    - Intrinsic momentum shifts displacement from equilibrium ($\boldsymbol{\xi}$).

Summary

- Dirac equation describes spin density.
- Spin is ordinary angular momentum due to rotational motion of an inertial medium.
- Elastic solid model of vacuum yields nonlinear Dirac eqn. with Hamiltonian = Total energy
- Kinetic energy ~ angular velocity · spin density.
- Potential energy ~ force density · displacement (force density ~ wave velocity rotation rate)

- Solid model also yields exclusion principle, interaction potentials, fermions, bosons, etc.

Publications

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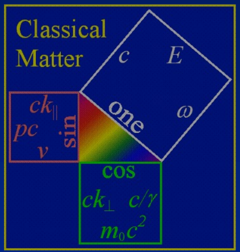
- More at: www.ClassicalMatter.org



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Conservation Law

Nonlinear Dirac equation:

$$\partial_t \psi = -c\,\gamma^5 \boldsymbol{\sigma} \cdot \nabla \psi - \mathbf{u} \cdot \nabla \psi - \frac{\mathrm{i}}{2}\left(\breve{w}_0\,\gamma^0 \psi + \mathbf{w} \cdot \boldsymbol{\sigma} \psi\right)$$

Multiply $\psi^\dagger$ and add adjoint:

$$\partial_t\left(\psi^\dagger \psi\right) = -\nabla \cdot \left(\psi^\dagger c\,\gamma^5 \boldsymbol{\sigma} \psi\right) - \mathbf{u} \cdot \nabla\left(\psi^\dagger \psi\right)$$

Magnitude $\psi^\dagger \psi$ is a conserved quantity.

Exclusion Principle & Potentials

- Wave superposition of "particles" A and B:

$$[\psi_A + \psi_B]^\dagger [\psi_A + \psi_B] = \psi_A^\dagger \psi_A + \psi_B^\dagger \psi_B + \psi_A^\dagger \psi_B + \psi_B^\dagger \psi_A$$

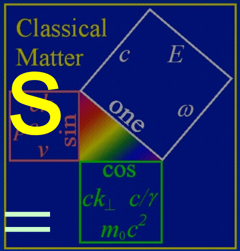
- Interference terms cancel for "independent" particles $\Longrightarrow$ exclusion principle:

$$\psi_A^\dagger \psi_B + \psi_B^\dagger \psi_A = 0$$

- Potentials derive from phase shifts introduced to maintain zero interference.

$$\psi_A^\dagger e^{-i\varphi_A} e^{i\varphi_B} \psi_B + c.c. = 0$$

Exclusion Principle & Potentials



Change of wave momentum (force): $\frac{d}{dt}P_i =$

$$\psi_A^\dagger\left((\partial_t\partial_i - \partial_i\partial_t)\varphi_A - c\gamma^5\sigma_j(\partial_i\partial_j - \partial_j\partial_i)\varphi_A\right)\psi_A$$

Multivalued phase $\Longrightarrow$ Non-commuting!

Electromagnetic variables:

$$e\mathbf{A} = -\hbar\nabla\varphi_A; \quad e\Phi = \hbar\partial_t\varphi_A$$

$$\rho_e = e\psi_A^\dagger\psi_A; \quad \mathbf{J} = e\psi_A^\dagger c\gamma^5\boldsymbol{\sigma}\psi_A$$

$$\mathbf{E} = -\frac{\hbar}{e}\nabla(\partial_t\varphi_A) + \frac{\hbar}{e}\partial_t(\nabla\varphi_A) = -\nabla\Phi - \partial_t\mathbf{A}$$

$$B_k = \epsilon_{kij}\partial_i A_j = -\frac{\hbar}{e}\epsilon_{kij}\partial_i\partial_j\varphi_A$$

$$\frac{d}{dt}\mathbf{P} = \rho_e\mathbf{E} + \mathbf{J}\times\mathbf{B}$$

Exclusion Principle & Potentials

Stokes' Law: $\oint \mathbf{A}\cdot d\boldsymbol{\ell} = \iint (\nabla \times \mathbf{A})\cdot d\mathbf{S}$

Macroscopic sytem:

Pure phase shift $\Rightarrow$ magnetic flux quantized:

$$\varphi_A = \left(m_\phi \phi - \omega t\right)$$

$$\mathbf{A} = -\frac{\hbar}{e}\nabla\varphi_A = -\frac{\hbar m_\phi}{er\ \sin\theta}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{\hbar m_\phi}{e} 2\pi\delta^{(\mathbf{2})}(x,y)$$

$$\oint \mathbf{A}\cdot d\boldsymbol{\ell} = -\frac{\hbar}{e}2\pi m_\phi \xrightarrow{m_\phi = 1/2} \frac{h}{2e}$$

c.f. Hagen Kleinert, *Multivalued Fields in Condensed Matter, Electromagnetism, and Gravitation* (Chapter 4).

Exclusion Principle & Potentials

Radially weighted phase shift from particle:

$$\varphi_A = (m_\phi \phi - \omega t)\left(\frac{e^2}{4\pi\epsilon_0 \hbar c}\right)\frac{c}{\omega r}$$

$$e\mathbf{A} = \left(\frac{e^2}{4\pi\epsilon_0 \hbar c}\right)\frac{c}{\omega r}\left\{\left(\frac{m_\phi}{r\ \sin\theta}\right)\hat{\phi} - \frac{m_\phi \phi - \omega t}{r^2}\hat{r}\right\}$$

Neglect radial term (no flux). Note $\hbar\omega = m_e c^2$

$$e\mathbf{B} = \left(\frac{e^2}{4\pi\epsilon_0 \hbar c}\right)\frac{c}{\omega}\left(\frac{m_\phi}{r^3\ \sin\theta}\right)\hat{\boldsymbol{\theta}}$$

Same flux as e^- dipole moment: $\mu_0 M/2r$:

$$\oint \mathbf{A}\cdot d\boldsymbol{\ell} = -\frac{m_\phi e}{2\epsilon_0 \omega r} \xrightarrow{m_\phi = 1/2} \frac{\mu_0 \hbar e}{4 m_e r}$$

Exclusion Principle & Potentials

Radially weighted phase shift:

$$\varphi_A = (m_\phi \phi - \omega t)\left(\frac{e^2}{4\pi\epsilon_0 \hbar c}\right)\frac{c}{\omega r}$$

Phase velocity: $v_\phi = \frac{\omega}{m_\phi}\, r \sin\theta \;\rightarrow 2\omega\, r \sin\theta$

Electric field (Note $\partial_t \varphi_A = -\mathbf{v}\cdot\nabla\varphi_A$):

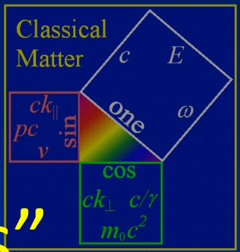
$$eE_i = \hbar\left(\partial_i v_j\right)\partial_j \varphi_A + \hbar v_j\left(\partial_i\, \partial_j - \partial_j\, \partial_i\right)\varphi_A$$

Only first term contributes ($v_\phi = 0$ at $r = 0$)

$$\mathbf{E} = \left(\frac{e}{4\pi\epsilon_0 r^2}\right)\hat{\mathbf{r}}$$

c.f. Herbert Jehle, Phys. Rev. 3(2):306-345, 1971

Matter & Antimatter



Assume vector spherical harmonic "particles".

Bispinor angular quantum number is half of vector angular quantum number.

Vector: $\ell = 2N + 1 \Longrightarrow$ Bispinor: $\ell = \frac{2N+1}{2}$

Odd parity $\Longrightarrow$ Distinct mirror image $\Longrightarrow$ Fermion

Vector: $\ell = 2N \Longrightarrow$ Bispinor: $\ell = N$

Even parity $\Longrightarrow$ Same mirror image $\Longrightarrow$ Boson

(bosons = antiparticles except W+/-)

Assumption valid except for W+/-.

Quantum Electrodynamics

2-Particle Lagrangian:

$$\mathfrak{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\,\partial_\mu - A_\mu) - m_A]\psi_A + (A \rightarrow B)$$

Apply Dirac equation to B: $\mathfrak{L} =$

$$\bar{\psi}_A[\gamma^\mu(\mathrm{i}\,\partial_\mu - A_\mu) - m_A]\psi_A + J^\mu A_\mu - J^\mu A_\mu$$

Green's 1st identity, vector identities, etc.:

$$\int J^\mu A_\mu = \int (E^2 - B^2) = -\frac{1}{2}\int F^{\mu\nu}F_{\mu\nu}$$

Factor of ½ due to counting variations twice:

$$\mathfrak{L} = \bar{\psi}_A[\gamma^\mu(\mathrm{i}\,\partial_\mu - A_\mu) - m_A]\psi_A$$
$$+\frac{1}{2}(E^2 - B^2) - J^\mu A_\mu$$